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Transmission rate of classical information through the thermalizing quantum channel

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Abstract

The transmission rate of classical information sent through the thermalizing quantum channel is considered. The entanglement-assisted capacity, the Holevo capacity and the capacity of the quantum dense coding are analytically obtained and their properties are investigated. It is found that due to the unlimited use of quantum entanglement, the entanglement-assisted capacity becomes much greater than any other capacities when the channel noise is very large and the signal power is very weak.

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1. Introduction

When we send some classical information to a distant receiver through a noisy quantum channel, the maximum rate of information transmission is determined by the classical capacity of a noisy quantum channel. In the classical information theory, the capacity of information transmission is uniquely determined by the maximum value of the Shannon mutual information of the noisy classical channel [1]. On the other hand, in the quantum information theory, the classical capacity of information transmission is determined by the several distinct quantities which depend on how quantum resources are used for information transmission. One of the most important resources is quantum entanglement. The Holevo capacity [2–4] seems to be the straightforward generalization of the Shannon capacity in the classical information theory to a noisy quantum channel. When a sender and receiver share a given entangled quantum state, the upper bound for the transmission rate of classical information is determined by the capacity of the quantum dense coding [5, 6]. Furthermore, when a sender and receiver can use quantum entanglement unlimitedly, the entanglement-assisted capacity determines the upper bound of the transmission rate of classical information sent through a noisy quantum channel [7, 8].

The one-shot Holevo capacity $C_H^{(1)}(\hat{\mathcal{L}})$ of a noisy quantum channel $\hat{\mathcal{L}}$ is given by

$$C_H^{(1)}(\hat{\mathcal{L}}) = \max_{\pi_k} \max_{\hat{\rho}_k \in \mathcal{H}} \left[S\left(\sum_k \pi_k \hat{\mathcal{L}} \hat{\rho}_k \right) - \sum_k \pi_k S(\hat{\mathcal{L}} \hat{\rho}_k) \right] \tag{1}$$

where the quantum state $\hat{\rho}_k$ is defined on the Hilbert space $\mathcal{H}$ and π_k is the prior probability of $\hat{\rho}_k$, and $S(\hat{\rho})$ is the von Neumann entropy of a quantum state $\hat{\rho}$. Then the Holevo capacity $C_H(\hat{\mathcal{L}})$ of a noisy quantum channel $\hat{\mathcal{L}}$ is defined by

$$C_H(\hat{\mathcal{L}}) = \lim_{n\to\infty} \frac{1}{n} C_H^{(n)}(\hat{\mathcal{L}}) \tag{2}$$

with

$$C_H^{(n)}(\hat{\mathcal{L}}) = \max_{\pi_k} \max_{\hat{\rho}_k \in \mathcal{H}^{\otimes n}} \left[S\left(\sum_k \pi_k \hat{\mathcal{L}}^{\otimes n} \hat{\rho}_k \right) - \sum_k \pi_k S(\hat{\mathcal{L}}^{\otimes n} \hat{\rho}_k) \right]. \tag{3}$$

It is an open problem whether the equality $C_H^{(n)}(\hat{\mathcal{L}}) = nC_H^{(1)}(\hat{\mathcal{L}})$ holds or not, though such an equality has been proved for the several quantum channel [10–12].

The entanglement-assisted capacity $C_E(\hat{\mathcal{L}})$ of a noisy quantum channel $\hat{\mathcal{L}}$ is given by [7, 8]

$$\begin{aligned} C_E(\hat{\mathcal{L}}) &= \max_{\hat{\rho} \in \mathcal{H}} I(\hat{\rho}, \hat{\mathcal{L}}) \\ &= \max_{\hat{\rho} \in \mathcal{H}} [S(\hat{\rho}) + S(\hat{\mathcal{L}}\hat{\rho}) - S_e(\hat{\rho}, \hat{\mathcal{L}})] \end{aligned} \tag{4}$$

where $I(\hat{\rho}, \hat{\mathcal{L}})$ is the quantum mutual information and $S_e(\hat{\rho}, \hat{\mathcal{L}})$ is the entropy exchange of the quantum channel $\hat{\mathcal{L}}$ [13–15]. The additivity of the entanglement-assisted capacity has been recently proved [16]. That is, the entanglement-assisted capacity satisfies the equality $C_E(\hat{\mathcal{L}}_1 \otimes \hat{\mathcal{L}}_2) = C_E(\hat{\mathcal{L}}_1) + C_E(\hat{\mathcal{L}}_2)$. When a sender and receiver share an entangled quantum state $\hat{W}$, the one-shot capacity $C_D^{(1)}(\hat{\mathcal{L}})$ of the quantum dense coding is given by

$$C_D^{(1)}(\hat{\mathcal{L}}) = \max_{\pi_k} \max_{\hat{\mathcal{E}}_k} \left[S\left(\sum_k \pi_k (\hat{\mathcal{L}}\hat{\mathcal{E}}_k \otimes \hat{\mathcal{I}}) \hat{W} \right) - \sum_k \pi_k S((\hat{\mathcal{L}}\hat{\mathcal{E}}_k \otimes \hat{\mathcal{I}}) \hat{W}) \right] \tag{5}$$

where the completely positive map $\hat{\mathcal{E}}_k$ describes the quantum encoding performed by the sender and π_k is the prior probability of $\hat{\mathcal{E}}_k$. When the unitary encoding is applied, the analytic expression of $C_D^{(1)}(\hat{\mathcal{L}})$ can be obtained [5, 6]. The capacity of the quantum dense coding is defined by $C_D(\mathcal{L}) = \lim_{n\to\infty} \left[C_D^{(n)}(\hat{\mathcal{L}})/n \right]$ in the same way as that of the Holevo capacity $C_H(\hat{\mathcal{L}})$. Since it is very difficult to obtain $C_H(\hat{\mathcal{L}})$ and $C_D(\hat{\mathcal{L}})$, we use the one-shot capacities $C_H^{(1)}(\hat{\mathcal{L}})$ and $C_D^{(1)}(\hat{\mathcal{L}})$ to investigate the properties of the entanglement-assisted capacity $C_E(\hat{\mathcal{L}})$. Hence we neglect the superscript '(1)' of the one-shot capacity in the rest of this paper.

In this paper, we consider the transmission rate of classical information sent through the thermalizing quantum channel. In section 2, we briefly review the mathematical method for calculating the von Neumann entropy, the Holevo capacity and the entanglement-assisted capacity of Gaussian quantum states and Gaussian quantum channels [17–19]. Gaussian quantum states are not only analytically tractable but also very important in quantum communication. For example, coherent and squeezed states including thermal noise are Gaussian. Gaussian quantum channels include the attenuation channel, the amplification channel and the thermalizing channel. In section 3, after summarizing the basic properties of the thermalizing quantum channel [20–22], we obtain the entanglement-assisted capacity $C_E(\hat{\mathcal{L}})$, the Holevo capacity $C_H(\hat{\mathcal{L}})$ and the capacity $C_D(\hat{\mathcal{L}})$ of the quantum dense coding. Using the results, we investigate the properties information transmission of the thermalizing quantum channel. In section 4, a summary is given.

2. Calculational method for Gaussian quantum states

2.1. The von Neumann entropy

The von Neumann entropy of an arbitrary Gaussian quantum state can be expressed in the simple and analytic form. This section briefly reviews the results obtained in [17–19]. Suppose an s-mode bosonic system with position operators $\hat{x}_1, \ldots, \hat{x}_s$ and momentum operators $\hat{p}_1, \ldots, \hat{p}_s$, satisfying the canonical commutation relation $[\hat{x}_j, \hat{p}_k] = \mathrm{i}\delta_{jk}$, where we set $\hbar = 1$. Real parameters $x_1, \ldots, x_s, p_1, \ldots, p_s$ are also introduced. Furthermore we use vector notations, $z = (x_1, \ldots, x_s, p_1, \ldots, p_s)^{\mathrm{T}}$ and $\hat{R} = (\hat{x}_1, \ldots, \hat{x}_s, \hat{p}_1, \ldots, \hat{p}_s)^{\mathrm{T}}$, where the symbol 'T' stands for the transposition of vectors and matrices. We define a unitary operator $\hat{V}(z)$ as

$$\hat{V}(z) = \exp[\mathrm{i}\hat{R}(z)] \tag{6}$$

with

$$\hat{R}(z) = z^{\mathrm{T}}\hat{R} = \hat{R}^{\mathrm{T}}z = \sum_{k=1}^{2s} z_k \hat{R}_k = \sum_{k=1}^{s} (x_k \hat{x}_k + p_k \hat{p}_k). \tag{7}$$

The operator $\hat{R}(z)$ obeys the commutation relation $[\hat{R}(z), \hat{R}(z')] = -\mathrm{i}\Delta(z, z')$ with $\Delta(z, z') = \sum_{k=1}^{s} (x'_k y_k - x_k y'_k)$. Thus the unitary operator $\hat{V}(z)$ satisfies the relation

$$\hat{V}(z)\hat{V}(z') = \hat{V}(z + z')\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\Delta(z,z')} = \hat{V}(z')\hat{V}(z)\,\mathrm{e}^{\mathrm{i}\Delta(z,z')}. \tag{8}$$

The characteristic function $C(z)$ of a quantum state $\hat{\rho}$ is defined by $C(z) = \mathrm{Tr}[\hat{V}(z)\hat{\rho}]$. Then a quantum state $\hat{\rho}$ is called Gaussian if and only if the characteristic function $C(z)$ of $\hat{\rho}$ becomes Gaussian, that is,

$$C(z) = \exp\left(\mathrm{i}m^{\mathrm{T}}z - \tfrac{1}{2}z^{\mathrm{T}}\alpha z\right) \tag{9}$$

where m is a $2s$-dimensional real vector and α is a $2s \times 2s$ real symmetric matrix which are given by

$$m = \mathrm{Tr}[\hat{R}\hat{\rho}] \qquad \alpha - \tfrac{1}{2}\mathrm{i}\Delta = \mathrm{Tr}[(\hat{R} - m)\hat{\rho}(\hat{R} - m)^{\mathrm{T}}]. \tag{10}$$

Here Δ is a $2s \times 2s$ real antisymmetric matrix defined by

$$\Delta = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix} \tag{11}$$

where I_s is a $s \times s$ unit matrix. The real symmetric matrix α must satisfy the inequalities $\alpha - \frac{1}{2}\mathrm{i}\Delta \geqslant 0$ and $\alpha + \frac{1}{2}\mathrm{i}\Delta \geqslant 0$ because of the Heisenberg uncertainty relation of position and momentum, $\Delta\hat{x}_k \Delta\hat{p}_k \geqslant 1/2$ ($k = 1, 2, \ldots, s$). The Gaussian quantum state $\hat{\rho}$ is completely determined by the two parameters m and α.

The von Neumann entropy $S(\hat{\rho})$ of an arbitrary Gaussian quantum state $\hat{\rho}$ can be calculated by the following formula [18]:

$$S(\hat{\rho}) = \tfrac{1}{2}\mathrm{Tr}\, G(-(\Delta^{-1}\alpha)^2) \tag{12}$$

where the function $G(x^2)$ is defined by

$$G(x^2) = \left(x + \tfrac{1}{2}\right)\ln\left(x + \tfrac{1}{2}\right) - \left(x - \tfrac{1}{2}\right)\ln\left(x - \tfrac{1}{2}\right). \tag{13}$$

In the paper, we measure the entropy and information in *nats*. As an example, let us consider a single mode bosonic system. In this case, the real symmetric matrix α and the real antisymmetric matrix Δ are given by

$$\alpha = \begin{pmatrix} \alpha_{xx} & \alpha_{xp} \\ \alpha_{xp} & \alpha_{pp} \end{pmatrix} \qquad \Delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

Since we obtain $(\Delta^{-1}\alpha)^2 = \left(\alpha_{xp}^2 - \alpha_{xx}\alpha_{pp}\right)I_2$, the von Neumann entropy of any Gaussian quantum state of a single mode bosonic system is given by

$$S(\hat{\rho}) = \left(\sqrt{\alpha_{xx}\alpha_{pp} - \alpha_{xp}^2} + \tfrac{1}{2}\right) \ln\left(\sqrt{\alpha_{xx}\alpha_{pp} - \alpha_{xp}^2} + \tfrac{1}{2}\right) - \left(\sqrt{\alpha_{xx}\alpha_{pp} - \alpha_{xp}^2} - \tfrac{1}{2}\right) \ln\left(\sqrt{\alpha_{xx}\alpha_{pp} - \alpha_{xp}^2} - \tfrac{1}{2}\right). \tag{15}$$

For the thermal equilibrium state with the average photon number $\bar{n}$, substituting $\alpha_{xx} = \alpha_{pp} = \bar{n} + \frac{1}{2}$ and $\alpha_{xp} = 0$ into this equation, we obtain the well-known result, namely, $S(\hat{\rho}) = (\bar{n}+1)\ln(\bar{n}+1) - \bar{n}\ln\bar{n}$.

2.2. The Holevo capacity

We consider a transmission of classical information represented by Gaussian quantum states [18]. In this case, a sender encodes classical information by applying the displacement operator $\hat{D}(\mu) = \exp\left(\sum_{k=1}^{s} \mu_k \hat{a}_k^\dagger - \mu_k^* \hat{a}_k\right)$ with probability $P(\mu)$ to a Gaussian quantum state $\hat{\rho}$, where $\hat{a}_k = (\hat{x}_k + \mathrm{i}\hat{p})/\sqrt{2}$ is a bosonic annihilation operator. The Holevo capacity C_H of such a quantum communication system is given by

$$C_H = \max_P \left[S\left(\int \mathrm{d}^{2s}\mu\, P(\mu)\hat{\rho}(\mu)\right) - \int \mathrm{d}^{2s}\mu\, P(\mu) S(\hat{\rho}(\mu)) \right] = \max_P S\left(\int \mathrm{d}^{2s}\mu\, P(\mu)\hat{\rho}(\mu)\right) - S(\hat{\rho}) \tag{16}$$

with $\hat{\rho}(\mu) = \hat{D}(\mu)\hat{\rho}\hat{D}^\dagger(\mu)$. It has been shown that the probability $P(\mu)$ can be restricted to a Gaussian probability distribution in the maximization of equation (16) [18]. The characteristic function $C_P(z)$ of the Gaussian probability distribution $P(\mu)$ is given by $C_P(z) = \exp(-z^{\mathrm{T}}\beta z/2)$ with β being a $2s \times 2s$ real symmetric matrix. Here we have ignored the linear term of z since such a term does not contribute to the final result. When we denote the average input power of the system as $\bar{m}$, the matrix β must satisfy the relation $\frac{1}{2}\,\mathrm{Tr}\beta = \bar{m}$. Thus using equation (12), we can express the Holevo capacity C_H of the quantum communication system with Gaussian quantum states in the following form:

$$C_H = \max_{\substack{\beta > 0 \\ (\frac{1}{2}\mathrm{Tr}\beta = \bar{m})}} \left[\tfrac{1}{2} G(-[\Delta^{-1}(\alpha+\beta)]^2)\right] - \tfrac{1}{2} G(-[\Delta^{-1}\alpha]^2). \tag{17}$$

For the thermal equilibrium state of a single mode bosonic system with the average photon number $\bar{n}$, the maximum value of equation (17) is attained when we set $\beta_{xx} = \beta_{pp} = \bar{m}$ and $\beta_{xp} = 0$, and hence the Holevo capacity becomes

$$C_H = (\bar{m}+\bar{n}+1)\ln(\bar{m}+\bar{n}+1) - (\bar{m}+\bar{n})\ln(\bar{m}+\bar{n}) - (\bar{n}+1)\ln(\bar{n}+1) + \bar{n}\ln\bar{n}. \tag{18}$$

2.3. The quantum mutual information

A quantum channel is described by a completely positive map $\hat{\mathcal{L}}$ which transforms an input quantum state $\hat{\rho}_{\mathrm{in}}$ into the output quantum state $\hat{\rho}_{\mathrm{out}} = \hat{\mathcal{L}}\hat{\rho}_{\mathrm{in}}$. A quantum channel $\hat{\mathcal{L}}$ is called Gaussian if and only if the output state $\hat{\rho}_{\mathrm{out}}$ becomes Gaussian for any Gaussian input state $\hat{\rho}_{\mathrm{in}}$. When the sender transmits a Gaussian quantum state to the receiver through a Gaussian quantum channel $\hat{\mathcal{L}}$, the capacity formula (17) is still valid if the real symmetric matrix α that appeared in the formula is replaced with that of the output Gaussian quantum state. Besides the von Neumann entropies $S(\hat{\rho}_{\mathrm{in}})$ and $S(\hat{\rho}_{\mathrm{out}})$ of the input and output quantum states, the entropy exchange $S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ is one of the most important quantities characterizing the quantum channel $\hat{\mathcal{L}}$ [13–15], which is given by

$$S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}) = S((\hat{\mathcal{L}} \otimes \hat{\mathcal{I}})|\Psi\rangle\langle\Psi|) \tag{19}$$

where $|\Psi\rangle$ is the purification of the input quantum state $\hat{\rho}_{\rm in}$, the partial trace of which is equal to the input quantum state $\hat{\rho}_{\rm in}$. It is important to note that although the purification is not unique, the entropy exchange $S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ is uniquely determined when the quantum channel $\hat{\mathcal{L}}$ and the input state $\hat{\rho}_{\rm in}$ are given [14]. For a Gaussian quantum channel $\hat{\mathcal{L}}$ and a Gaussian input state $\hat{\rho}_{\rm in}$, the entropy exchange $S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ is calculated by the following formula [19]:

$$S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}) = \tfrac{1}{2}{\rm Tr}\, G\big(- \big(\Delta_{\rm ex}^{-1}\alpha_{\rm ex}\big)^2\big) \tag{20}$$

with

$$\alpha_{\rm ex} = \begin{pmatrix} \alpha_{\rm out} & K\beta \\ \beta^{\rm T}K^{\rm T} & \alpha_{\rm in} \end{pmatrix} \qquad \Delta_{\rm ex} = \begin{pmatrix} \Delta & 0 \\ 0 & -\Delta \end{pmatrix} \tag{21}$$

where $\alpha_{\rm in}$ and $\alpha_{\rm out}$ are the real symmetric matrices appeared in the characteristic functions of the input and output Gaussian quantum states, $\hat{\rho}_{\rm in}$ and $\hat{\rho}_{\rm out} = \hat{\mathcal{L}}\hat{\rho}_{\rm in}$, and the matrix β is given by

$$\beta = \Delta\sqrt{-(\Delta^{-1}\alpha_{\rm in})^2 - \tfrac{1}{4}I}. \tag{22}$$

The matrix K appearing in equation (21) is determined by the following relation of the Gaussian quantum channel $\hat{\mathcal{L}}$ [19]:

$$\hat{\mathcal{L}}^{\dagger}\hat{V}(z) = \hat{V}(K^{\rm T}z) f(z) \tag{23}$$

where $f(z)$ is some Gaussian function.

For a given quantum channel $\hat{\mathcal{L}}$ and input quantum state $\hat{\rho}_{\rm in}$, the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ is defined by [13]

$$I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}) = S(\hat{\rho}_{\rm in}) + S(\hat{\rho}_{\rm out}) - S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}) \tag{24}$$

with $\hat{\rho}_{\rm out} = \hat{\mathcal{L}}\hat{\rho}_{\rm in}$. It has been shown that when the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ for the Gaussian quantum channel $\hat{\mathcal{L}}$ is maximized with respect to the input state $\hat{\rho}_{\rm in}$, it is enough to consider Gaussian quantum states [19]. Since the von Neumann entropy of any Gaussian quantum state is completely determined by the real symmetric matrix $\alpha_{\rm in}$ appearing in the characteristic function, the maximum value of the quantum mutual information can be expressed as

$$\max_{\hat{\rho}_{\rm in}} I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}) = \tfrac{1}{2}\max_{\alpha_{\rm in}} {\rm Tr}[G(-(\Delta^{-1}\alpha_{\rm in})^2) + G(-(\Delta^{-1}\alpha_{\rm out})^2) - G\big(- \big(\Delta_{\rm ex}^{-1}\alpha_{\rm ex})^2\big)\big] \tag{25}$$

where the maximization on the right-hand side must be performed under the constraint of the positivity of $\alpha_{\rm in}$ and the uncertainty relation $\alpha_{\rm in} \pm {\rm i}\frac{1}{2}\Delta \geqslant 0$. The maximum value of quantum mutual information $\max_{\hat{\rho}_{\rm in}} I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}})$ is equal to the entanglement-assisted capacity $C_E(\hat{\mathcal{L}})$ of classical information sent through the Gaussian quantum channel $\hat{\mathcal{L}}$. The derivations of the results are given in [18, 19].

3. Capacity of the thermalizing quantum channel

3.1. The thermalizing quantum channel

Before obtaining the channel capacities, we summarize the basic properties of the thermalizing quantum channel [20]. A completely positive map $\hat{\mathcal{L}}_{\bar{n}}$ describing the thermalizing quantum channel is given by

$$\hat{\mathcal{L}}_{\bar{n}}\hat{\rho} = \int \frac{{\rm d}^2\beta}{\pi\bar{n}}\,{\rm e}^{-|\beta|^2/\bar{n}}\,\hat{D}(\beta)\hat{\rho}\hat{D}^{\dagger}(\beta) \tag{26}$$

where $\hat{D}(\beta)$ is the displacement operator and the parameter $\bar{n}$ stands for the average photon number of the channel noise. Of course, $\hat{\mathcal{L}}_{\bar{n}=0}$ reduces to an identity map $\hat{\mathcal{I}}$ (a noiseless quantum channel). The thermalizing quantum channel $\hat{\mathcal{L}}_{\bar{n}}$ satisfies

$$\hat{\mathcal{L}}_{\bar{n}}[D^{\dagger}(\alpha)\hat{\rho}\hat{D}^{\dagger}(\alpha)] = D^{\dagger}(\alpha)[\hat{\mathcal{L}}_{\bar{n}}\hat{\rho}]\hat{D}^{\dagger}(\alpha) \tag{27}$$

and

$$\hat{\mathcal{L}}_{\bar{n}}\left(\int \frac{\mathrm{d}^2\alpha}{\pi\bar{m}}\,\mathrm{e}^{-|\alpha|^2/\bar{m}}\,\hat{D}(\alpha)\hat{\rho}\hat{D}^{\dagger}(\alpha)\right) = \hat{\mathcal{L}}_{\bar{m}+\bar{n}}\hat{\rho} \tag{28}$$

which is equivalent to the relation $\hat{\mathcal{L}}_{\bar{m}}\hat{\mathcal{L}}_{\bar{n}} = \hat{\mathcal{L}}_{\bar{m}+\bar{n}}$. The thermalizing quantum channel appears in the quantum communication systems with the Gaussian noise [20] and the quantum teleportation system [22]. The characteristic function $C_{\mathrm{out}}(z)$ of the output state of the thermalizing quantum channel is calculated to be

$$C_{\mathrm{out}}(z) = \mathrm{Tr}[\hat{V}(z)\hat{\mathcal{L}}_{\bar{n}}\hat{\rho}_{\mathrm{in}}] = \exp\left(-\tfrac{1}{2}\bar{n}|z|^2\right)C_{\mathrm{in}}(z) \tag{29}$$

where $C_{\mathrm{in}}(z)$ is the characteristic function of the input quantum state $\hat{\rho}_{\mathrm{in}}$. This result shows that the function $f(z)$ and the matrix K characterizing the Gaussian quantum channel (see equation (23)) are given by

$$f(z) = \exp\left(-\tfrac{1}{2}\bar{n}|z|^2\right) \qquad K = I. \tag{30}$$

When $\hat{\rho}_{\mathrm{in}}$ is the Gaussian quantum state with the characteristic function $C_{\mathrm{in}}(z) = \exp(-z^{\mathrm{T}}\alpha_{\mathrm{in}}z/2)$, we obtain $C_{\mathrm{out}}(z) = \exp(-z^{\mathrm{T}}\alpha_{\mathrm{out}}z/2)$ with $\alpha_{\mathrm{out}} = \alpha_{\mathrm{in}} + \bar{n}I$, where we have ignored the vector m of the first moment of the operator $\hat{R}$ since it is not important when we investigate the channel capacities. Hence using the formula (12), we obtain the von Neumann entropies of the input and output states of the single mode bosonic system in the thermalizing quantum channel $\hat{\mathcal{L}}_{\bar{n}}$

$$S(\hat{\rho}_{\mathrm{in}}) = \left(f_0(\alpha) + \tfrac{1}{2}\right)\ln\left(f_0(\alpha) + \tfrac{1}{2}\right) - \left(f_0(\alpha) - \tfrac{1}{2}\right)\ln\left(f_0(\alpha) - \tfrac{1}{2}\right) \tag{31}$$

and

$$S(\hat{\rho}_{\mathrm{out}}) = \left(f_{\bar{n}}(\alpha) + \tfrac{1}{2}\right)\ln\left(f_{\bar{n}}(\alpha) + \tfrac{1}{2}\right) - \left(f_{\bar{n}}(\alpha) - \tfrac{1}{2}\right)\ln\left(f_{\bar{n}}(\alpha) - \tfrac{1}{2}\right) \tag{32}$$

where the function $f_{\bar{n}}(\alpha)$ is given by

$$f_{\bar{n}}(\alpha) = \sqrt{(\alpha_{xx} + \bar{n})(\alpha_{pp} + \bar{n}) - \alpha_{xp}^2}. \tag{33}$$

The entropy exchange $S_e(\hat{\rho}_{\mathrm{in}}, \hat{\mathcal{L}}_{\bar{n}})$ of the thermalizing quantum channel is derived from equations (20)–(22)

$$\begin{aligned} S_e(\hat{\rho}_{\mathrm{in}}, \hat{\mathcal{L}}_{\bar{n}}) = &\left(\lambda_+ + \tfrac{1}{2}\right)\ln\left(\lambda_+ + \tfrac{1}{2}\right) - \left(\lambda_+ - \tfrac{1}{2}\right)\ln\left(\lambda_+ - \tfrac{1}{2}\right) \\ &+ \left(\lambda_- + \tfrac{1}{2}\right)\ln\left(\lambda_- + \tfrac{1}{2}\right) - \left(\lambda_- - \tfrac{1}{2}\right)\ln\left(\lambda_- - \tfrac{1}{2}\right) \end{aligned} \tag{34}$$

with

$$\lambda_{\pm}^2 = \tfrac{1}{2}\left[\bar{n}(\alpha_{xx} + \alpha_{pp} + \bar{n}) + \tfrac{1}{2} \pm \sqrt{\bar{n}^2(\alpha_{xx} + \alpha_{pp} + \bar{n})^2 - \bar{n}^2\left(4\alpha_{xx}\alpha_{pp} - 4\alpha_{xp}^2 - 1\right)}\right]. \tag{35}$$

Using these results, we can obtain the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$, the Holevo capacity $C_H(\hat{\mathcal{L}}_{\bar{n}})$ and the quantum dense coding capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$ of the thermalizing quantum channel.

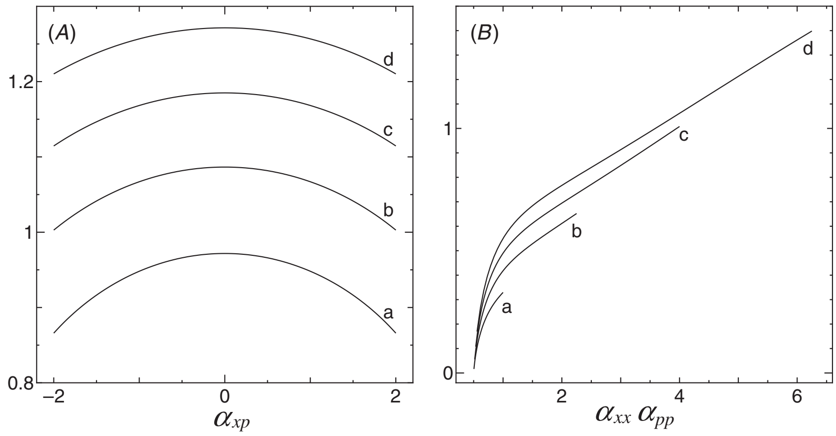

Figure 1. The dependence of the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ of the real symmetric matrix α appeared in the characteristic function. In (*A*), the dependence on the off-diagonal element α_{xp} is plotted, where (a) $(\alpha_{xx}, \alpha_{pp}, \bar{n}) = (3.0, 3.0, 3.0)$, (b) $(\alpha_{xx}, \alpha_{pp}, \bar{n}) = (4.0, 3.0, 3.0)$, (c) $(\alpha_{xx}, \alpha_{pp}, \bar{n}) = (5.0, 3.0, 3.0)$, and (d) $(\alpha_{xx}, \alpha_{pp}, \bar{n}) = (6.0, 3.0, 3.0)$. In (*B*) the dependence on the product $\alpha_{xx}\alpha_{pp}$ in the case of $\alpha_{xp} = 0$ and $\bar{n} = 3.0$ is plotted, where: a, $\alpha_{xx} + \alpha_{pp} = 2.0$; b, $\alpha_{xx} + \alpha_{pp} = 3.0$; c, $\alpha_{xx} + \alpha_{pp} = 4.0$; and d, $\alpha_{xx} + \alpha_{pp} = 5.0$. In the figures, the quantum mutual information is measured in bits.

3.2. The entanglement-assisted capacity

We obtain the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$ of the thermalizing quantum channel $\hat{\mathcal{L}}_{\bar{n}}$ given by equation (26), which is the maximum value of the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ with respect to the input Gaussian state $\hat{\rho}_{\rm in}$. The quantum mutual information of the thermalizing quantum channel is given by

$$I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}}) = S(\hat{\rho}_{\rm in}) + S(\hat{\rho}_{\rm out}) - S_e(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}}) \tag{36}$$

where the entropies appeared on the right-hand side are given by equations (31), (32) and (34). The maximization with respect to the input Gaussian state $\hat{\rho}_{\rm in}$ is equivalent to that with respect to the real symmetric matrix $\alpha_{\rm in}$ that appeared in the characteristic function of $\hat{\rho}_{\rm in}$. We first consider the variation of the off-diagonal element α_{xp} of the matrix $\alpha_{\rm in}$. It is easy to see that the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ is the function of α_{xp}^2. The function $f_{\bar{n}}(\alpha)$ decreases monotonically with α_{xp}^2 and the parameter $\lambda_\pm$ satisfies the inequality $\pm\partial\lambda_\pm\big/\partial\alpha_{xp}^2 > 0$. Using these results, we see that the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ decreases monotonically with α_{xp}^2. Hence the maximum value of $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ is attained at $\alpha_{xp} = 0$. The dependence of the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ on the off-diagonal element α_{xp} is plotted in figure 1(*A*). Next we consider the variation of the diagonal elements α_{xx} and α_{pp} of the matrix $\alpha_{\rm in}$. Suppose here that the average photon number of the input Gaussian state $\hat{\rho}_{\rm in}$ is $\bar{m}$, that is, $\bar{m} = {\rm Tr}[\hat{a}^\dagger\hat{a}\hat{\rho}_{\rm in}]$. In this case, the diagonal elements α_{xx} and α_{pp} must satisfy the equality $\alpha_{xx} + \alpha_{pp} = 2\bar{m} + 1$. Since the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ is the symmetric function of α_{xx} and α_{pp}, it becomes the function of $\alpha_{xx} + \alpha_{pp}$ and $\alpha_{xx}\alpha_{pp}$. Furthermore, taking into account the condition that $\alpha_{xx} + \alpha_{pp} = 2\bar{m} + 1$, the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ is a function only of the product $\alpha_{xx}\alpha_{pp}$, the range of which is $0 < \alpha_{xx}\alpha_{pp} \leqslant [(\alpha_{xx} + \alpha_{pp})/2]^2 = (\bar{m} + 1/2)^2$. It is found that the quantum mutual information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ increases monotonically with $\alpha_{xx}\alpha_{pp}$. Thus the maximum value is attained at $\alpha_{xx} = \alpha_{pp} = \bar{m} + 1/2$. The dependence of the quantum mutual information on the product $\alpha_{xx}\alpha_{pp}$ is plotted in figure 1(*B*).

Summarizing the result, the real symmetric matrix α_{in} that maximizes the quantum mutual information $I(\hat{\rho}_{\text{in}}, \hat{\mathcal{L}}_{\bar{n}})$ is given by

$$\alpha_{\text{in}} = \begin{pmatrix} \bar{m} + \frac{1}{2} & 0 \\ 0 & \bar{m} + \frac{1}{2} \end{pmatrix}. \tag{37}$$

This indicates that the optimum input Gaussian state $\hat{\rho}_{\text{in}}$ is the thermal equilibrium state with the average photon number $\bar{m}$,

$$\hat{\rho}_{\text{in}} = \frac{1}{\bar{m}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{m}}{\bar{m}+1} \right)^n |n\rangle\langle n| \equiv \hat{\rho}_{\bar{m}} \tag{38}$$

the purification $|\Psi_{\bar{m}}\rangle$ of which is the two-mode squeezed-vacuum state with squeezing parameter r satisfying the relation $\tanh r = \sqrt{\bar{m}/(\bar{m}+1)}$, that is,

$$|\Psi_{\bar{m}}\rangle = \frac{1}{\sqrt{\bar{m}+1}} \sum_{n=0}^{\infty} \left(\frac{\bar{m}}{\bar{m}+1} \right)^{n/2} |n\rangle \otimes |n\rangle. \tag{39}$$

Hence we finally obtain the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$ of the quantum thermalizing channel $\hat{\mathcal{L}}_{\bar{n}}$

$$\begin{aligned} C_E(\hat{\mathcal{L}}_{\bar{n}}) &= S(\hat{\rho}_{\bar{m}}) + S(\hat{\rho}_{\bar{m}+\bar{n}}) - S\left((\hat{\mathcal{L}}_{\bar{n}} \otimes \hat{\mathcal{I}}) |\Psi_{\bar{m}}\rangle\langle\Psi_{\bar{m}}| \right) \\ &= (\bar{m}+1)\ln(\bar{m}+1) - \bar{m}\ln\bar{m} \\ &\quad + (\bar{m}+\bar{n}+1)\ln(\bar{m}+\bar{n}+1) - (\bar{m}+\bar{n})\ln(\bar{m}+\bar{n}) \\ &\quad - \left(\frac{\bar{\omega}+\bar{n}+1}{2} \right) \ln\left(\frac{\bar{\omega}+\bar{n}+1}{2} \right) + \left(\frac{\bar{\omega}+\bar{n}-1}{2} \right) \ln\left(\frac{\bar{\omega}+\bar{n}-1}{2} \right) \\ &\quad - \left(\frac{\bar{\omega}-\bar{n}+1}{2} \right) \ln\left(\frac{\bar{\omega}-\bar{n}+1}{2} \right) + \left(\frac{\bar{\omega}-\bar{n}-1}{2} \right) \ln\left(\frac{\bar{\omega}-\bar{n}-1}{2} \right) \end{aligned} \tag{40}$$

where the parameter $\bar{\omega}$ is given by $\bar{\omega} = \sqrt{(\bar{n}+1)^2 + 4\bar{m}\bar{n}}$. For $\bar{n} \ll 1$, the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$ becomes

$$C_E(\hat{\mathcal{L}}_{\bar{n}}) \approx 2\left[(\bar{m}+1)\ln(\bar{m}+1) - \bar{m}\ln\bar{m}\right] \tag{41}$$

and for $\bar{n} \gg 1$

$$C_E(\hat{\mathcal{L}}_{\bar{n}}) \approx \frac{\bar{m}(\bar{m}+1)}{\bar{n}} \ln\left(\frac{\bar{m}+1}{\bar{m}} \right). \tag{42}$$

The entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$ of the thermalizing quantum channel is plotted in figure 2. For example, when $\bar{m} = 10.0$ and $\bar{n} = 1.0$, the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$ becomes 3.50 bits.

3.3. The Holevo capacity

When the sender and the receiver do not use the quantum entanglement as a resource of the quantum communication system, the upper bound of the transmission rate of classical information is given by the Holevo capacity $C_H(\hat{\mathcal{L}})$. For the thermalizing quantum channel $\hat{\mathcal{L}}_{\bar{n}}$, when the sender transmits the coherent state $|\alpha\rangle$ with the Gaussian probability distribution $P(\alpha) = (1/\pi\bar{m})\,\mathrm{e}^{-|\alpha|^2/\bar{m}}$, where $\bar{m}$ stands for the average photon number of the input [20, 21], the Holevo capacity is attained. Note that in this case, the averaged quantum state $\int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|$ is equal to the input Gaussian state that maximizes the quantum mutual

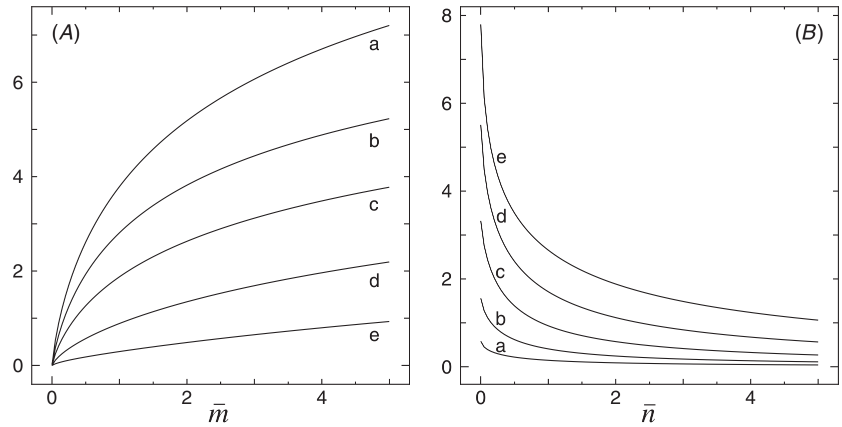

Figure 2. The entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$. ($A$) Shows the dependence on the input power $\bar{m}$ for: a, $\bar{n} = 0.01$; b, $\bar{n} = 0.12$; c, $\bar{n} = 0.4$; d, $\bar{n} = 1.5$; e, $\bar{n} = 6.0$. (B) Shows the dependence on the noise power $\bar{n}$ for: a, $\bar{m} = 0.05$; b, $\bar{m} = 0.2$; c, $\bar{m} = 0.7$; d, $\bar{m} = 2.0$; e, $\bar{m} = 5.0$. In the figures, the information is measured in bits.

information $I(\hat{\rho}_{\rm in}, \hat{\mathcal{L}}_{\bar{n}})$ of the thermalizing quantum channel $\hat{\mathcal{L}}_{\bar{n}}$. Then the Holevo capacity is given by

$$\begin{aligned} C_H(\hat{\mathcal{L}}_{\bar{n}}) &= S\left(\int \mathrm{d}^2\alpha\, P(\alpha)\hat{\mathcal{L}}_{\bar{n}}|\alpha\rangle\langle\alpha|\right) - \int \mathrm{d}^2\alpha\, P(\alpha) S(\hat{\mathcal{L}}_{\bar{n}}|\alpha\rangle\langle\alpha|) \\ &= S\left(\int \mathrm{d}^2\alpha\, P(\alpha)\hat{\mathcal{L}}_{\bar{n}}|\alpha\rangle\langle\alpha|\right) - S(\hat{\mathcal{L}}_{\bar{n}}|0\rangle\langle 0|) \\ &= S(\hat{\mathcal{L}}_{\bar{m}+\bar{n}}|0\rangle\langle 0|) - S(\hat{\mathcal{L}}_{\bar{n}}|0\rangle\langle 0|) \end{aligned} \tag{43}$$

where we have used equation (28). It is easy to obtain the characteristic function of the quantum state $\hat{\mathcal{L}}_{\bar{m}+\bar{n}}|0\rangle\langle 0|$

$$\mathrm{Tr}[\hat{V}(z)\hat{\mathcal{L}}_{\bar{m}+\bar{n}}|0\rangle\langle 0|] = \exp\left[-\tfrac{1}{2}\left(\bar{m}+\bar{n}+\tfrac{1}{2}\right)|z|^2\right]. \tag{44}$$

Hence the Holevo capacity $C_H(\hat{\mathcal{L}}_{\bar{n}})$ of the quantum thermalizing channel is obtained from equation (12)

$$C_H(\hat{\mathcal{L}}_{\bar{n}}) = (\bar{m}+\bar{n}+1)\ln(\bar{m}+\bar{n}+1) - (\bar{m}+\bar{n})\ln(\bar{m}+\bar{n}) - (\bar{n}+1)\ln(\bar{n}+1) + \bar{n}\ln\bar{n}. \tag{45}$$

When $\bar{m} = 10.0$ and $\bar{n} = 1.0$, the Holevo capacity $C_H(\hat{\mathcal{L}}_{\bar{n}})$ becomes 2.97 bits.

We now compare the Holevo capacity $C_H(\hat{\mathcal{L}}_{\bar{n}})$ with the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$. In the noiseless limit ($\bar{n} \to 0$), we easily find that

$$\lim_{\bar{n}\to 0} \frac{C_E(\hat{\mathcal{L}}_{\bar{n}})}{C_H(\hat{\mathcal{L}}_{\bar{n}})} = 2. \tag{46}$$

This result means that when the sender and the receiver use the quantum entanglement, the transmission rate of classical information becomes double. On the other hand, in the very noisy limit ($\bar{n} \to \infty$), we obtain

$$\lim_{\bar{n}\to\infty} \frac{C_E(\hat{\mathcal{L}}_{\bar{n}})}{C_H(\hat{\mathcal{L}}_{\bar{n}})} = (\bar{m}+1)\ln\left(\frac{\bar{m}+1}{\bar{m}}\right). \tag{47}$$

In this case, the quantum entanglement is more effective for the information transmission as the input power becomes smaller. In particular, we have

$$\lim_{\bar{m}\to\infty}\left[\lim_{\bar{n}\to\infty} \frac{C_E(\hat{\mathcal{L}}_{\bar{n}})}{C_H(\hat{\mathcal{L}}_{\bar{n}})}\right] = 1 \qquad \lim_{\bar{m}\to 0}\left[\lim_{\bar{n}\to\infty} \frac{C_E(\hat{\mathcal{L}}_{\bar{n}})}{C_H(\hat{\mathcal{L}}_{\bar{n}})}\right] = \infty. \tag{48}$$

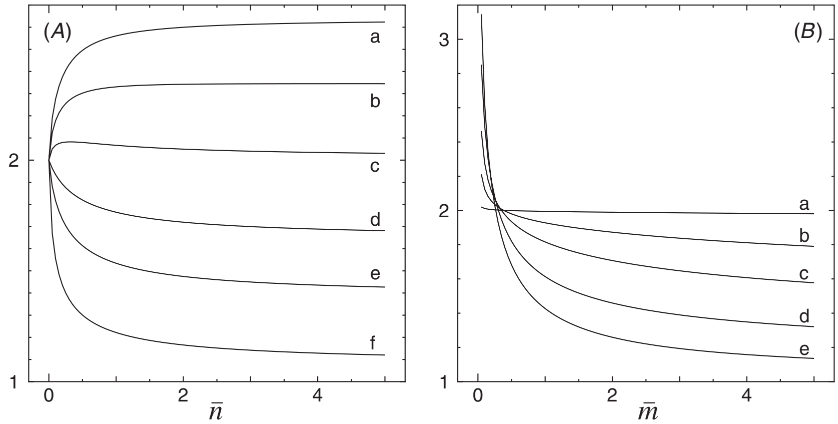

Figure 3. The dependence of the ratio $C_E(\hat{\mathcal{L}}_{\bar{n}})/C_H(\hat{\mathcal{L}}_{\bar{n}})$ on the parameters $\bar{m}$ and $\bar{n}$. (*A*) We set: a, $\bar{m} = 0.1$; b, $\bar{m} = 0.15$; c, $\bar{m} = 0.25$; d, $\bar{m} = 0.5$; e, $\bar{m} = 1.0$; f, $\bar{m} = 6.0$ and in (*B*) we set: a, $\bar{n} = 0.001$; b, $\bar{n} = 0.025$; c, $\bar{n} = 0.1$; d, $\bar{n} = 0.5$; e, $\bar{n} = 5.0$.

The ratio $C_E(\hat{\mathcal{L}}_{\bar{n}})/C_H(\hat{\mathcal{L}}_{\bar{n}})$ is plotted in figure 3. For $\bar{m} = 10.0$ and $\bar{n} = 1.0$, we obtain $C_E(\hat{\mathcal{L}}_{\bar{n}})/C_H(\hat{\mathcal{L}}_{\bar{n}}) = 1.18$. The condition that the ratio $C_E(\hat{\mathcal{L}}_{\bar{n}})/C_H(\hat{\mathcal{L}}_{\bar{n}})$ is greater than two for any value $\bar{n}$ of the channel noise is for the input power $\bar{m}$ to satisfy the inequality $\bar{m} < 0.255$. It is found from the result that the quantum entanglement is more effective for the classical information transmission through the thermalizing quantum channel as the input power $\bar{m}$ becomes smaller and the channel noise $\bar{n}$ larger.

3.4. The quantum dense coding capacity

Classical information can be transmitted through a noisy quantum channel by means of the quantum dense coding of continuous variables, which is usually performed by sharing the two-mode squeezed-vacuum state [23–25]

$$|\Psi_{\rm SV}\rangle = \frac{1}{\sqrt{\bar{n}_{\rm sq}+1}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}_{\rm sq}}{\bar{n}_{\rm sq}+1}\right)^{n/2} |n\rangle \otimes |n\rangle \tag{49}$$

where $\bar{n}_{\rm sq}$ represents the average photon number of each mode. There are two situations that we should consider. One is the case that the sender and the receiver share the ideal two-mode squeezed-vacuum state and the encoded mode is sent through a noisy quantum channel. The other is the case that the sender and the receiver share the non-ideal two-mode squeezed-vacuum state under the influence of the external environment. It is clear that the information transmission rate in the former case is greater than that in the latter case. In this section, we consider the former case to compare the capacity with the entanglement-assisted capacity $C_E(\hat{\mathcal{L}}_{\bar{n}})$. Suppose that the sender encodes the information by applying the displacement operator $\hat{D}(\alpha)$ with probability $P(\alpha) = (1/\pi\bar{m})\,{\rm e}^{-|\alpha|^2/\bar{m}}$. Then the capacity of the quantum dense coding of continuous variables is given by

$$C_D(\hat{\mathcal{L}}_{\bar{n}}) = S\left((\hat{\mathcal{L}}_{\bar{n}} \otimes \hat{\mathcal{I}}) \int {\rm d}^2\alpha\, P(\alpha)(\hat{D}(\alpha)\otimes\hat{I})|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|(\hat{D}(\alpha)\otimes\hat{I})^\dagger\right) - \int {\rm d}^2\alpha\, P(\alpha) S((\hat{\mathcal{L}}_{\bar{n}} \otimes \hat{\mathcal{I}})(\hat{D}(\alpha)\otimes\hat{I})|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|(\hat{D}(\alpha)\otimes\hat{I})^\dagger). \tag{50}$$

We obtain from equation (28)

$$C_D(\hat{\mathcal{L}}_{\bar{n}}) = S((\hat{\mathcal{L}}_{\bar{m}+\bar{n}} \otimes \hat{\mathcal{I}})|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|) - S((\hat{\mathcal{L}}_{\bar{n}} \otimes \hat{\mathcal{I}})|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|). \tag{51}$$

To calculate the capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$, we introduce the following function:

$$F(\bar{n}_A, \bar{n}_B) = S((\hat{\mathcal{L}}_{\bar{n}_A} \otimes \hat{\mathcal{L}}_{\bar{n}_B})|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|) \tag{52}$$

in terms of which the capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$ of the quantum dense coding can be expressed as $C_D(\hat{\mathcal{L}}_{\bar{n}}) = F(\bar{m}+\bar{n}, 0) - F(\bar{n}, 0)$. If the two-mode squeezed-vacuum state is shared through the thermalizing quantum channel, the capacity of the quantum dense coding is modified as $C_D(\hat{\mathcal{L}}_{\bar{n}}) = F(\bar{m}+\bar{n}, \bar{n}) - F(\bar{n}, \bar{n})$. It is easy to see that the characteristic function $C_{\bar{n}_A\bar{n}_B}(z)$ of the quantum state $\left(\hat{\mathcal{L}}_{\bar{n}_A} \otimes \hat{\mathcal{L}}_{\bar{n}_B}\right)|\Psi_{\rm SV}\rangle\langle\Psi_{\rm SV}|$ is given by

$$C_{\bar{n}_A\bar{n}_B}(z) = \exp\left(-\tfrac{1}{2}\bar{n}_A|z_A|^2 - \tfrac{1}{2}\bar{n}_B|z_B|^2\right) C_{\rm SV}(z) \tag{53}$$

with $z = (x_A, x_B, p_A, p_B)^{\rm T}$ and $z_A = (x_A, p_A)^{\rm T}$ $[z_B = (x_B, p_B)^{\rm T}]$. The characteristic function $C_{\rm SV}(z)$ of the two-mode squeezed-vacuum state $|\Psi_{\rm SV}\rangle$ is calculated to be

$$C_{\rm SV}(z) = \exp\left(-\tfrac{1}{2}z^{\rm T}\alpha_{\rm SV}z\right) \tag{54}$$

with

$$\alpha_{\rm SV} = \begin{pmatrix} \frac{1}{2}\cosh 2r & \frac{1}{2}\sinh 2r & 0 & 0 \\ \frac{1}{2}\sinh 2r & \frac{1}{2}\cosh 2r & 0 & 0 \\ 0 & 0 & \frac{1}{2}\cosh 2r & -\frac{1}{2}\sinh 2r \\ 0 & 0 & -\frac{1}{2}\sinh 2r & \frac{1}{2}\cosh 2r \end{pmatrix} \tag{55}$$

where $\cosh r = \sqrt{\bar{n}_{\rm sq}+1}$ and $\sinh r = \sqrt{\bar{n}_{\rm sq}}$. Hence the characteristic function $C_{\bar{n}_A\bar{n}_B}(z)$ becomes

$$C_{\bar{n}_A\bar{n}_B}(z) = \exp\left(-\tfrac{1}{2}z^{\rm T}\alpha_{AB}z\right) \tag{56}$$

with

$$\alpha_{AB} = \begin{pmatrix} \frac{1}{2}\cosh r + \bar{n}_A & \frac{1}{2}\sinh r & 0 & 0 \\ \frac{1}{2}\sinh r & \frac{1}{2}\cosh r + \bar{n}_B & 0 & 0 \\ 0 & 0 & \frac{1}{2}\cosh r + \bar{n}_A & -\frac{1}{2}\sinh r \\ 0 & 0 & -\frac{1}{2}\sinh r & \frac{1}{2}\cosh r + \bar{n}_B \end{pmatrix}. \tag{57}$$

Using formula (12), we can obtain the function $F(\bar{n}_A, \bar{n}_B)$ as

$$\begin{aligned} F(\bar{n}_A, \bar{n}_B) = {} & \left(\frac{\bar{\nu}_{AB} + \bar{n}_A - \bar{n}_B + 1}{2}\right) \ln\left(\frac{\bar{\nu}_{AB} + \bar{n}_A - \bar{n}_B + 1}{2}\right) \\ & - \left(\frac{\bar{\nu}_{AB} + \bar{n}_A - \bar{n}_B - 1}{2}\right) \ln\left(\frac{\bar{\nu}_{AB} + \bar{n}_A - \bar{n}_B - 1}{2}\right) \\ & + \left(\frac{\bar{\nu}_{AB} - \bar{n}_A + \bar{n}_B + 1}{2}\right) \ln\left(\frac{\bar{\nu}_{AB} - \bar{n}_A + \bar{n}_B + 1}{2}\right) \\ & - \left(\frac{\bar{\nu}_{AB} - \bar{n}_A + \bar{n}_B - 1}{2}\right) \ln\left(\frac{\bar{\nu}_{AB} - \bar{n}_A + \bar{n}_B - 1}{2}\right) \end{aligned} \tag{58}$$

where the parameter $\bar{\nu}_{AB}$ is given by

$$\bar{\nu}_{AB} = \sqrt{(\bar{n}_A + \bar{n}_B + 1)^2 + 4\bar{n}_{\rm sq}(\bar{n}_A + \bar{n}_B)}. \tag{59}$$

The capacity of the quantum dense coding becomes $C_D(\hat{\mathcal{L}}_{\bar{n}}) = F(\bar{m}+\bar{n}, 0) - F(\bar{n}, 0)$. It is easy to see that if there is no squeezing ($\bar{n}_{\rm sq} = 0$), the quantum dense coding capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$

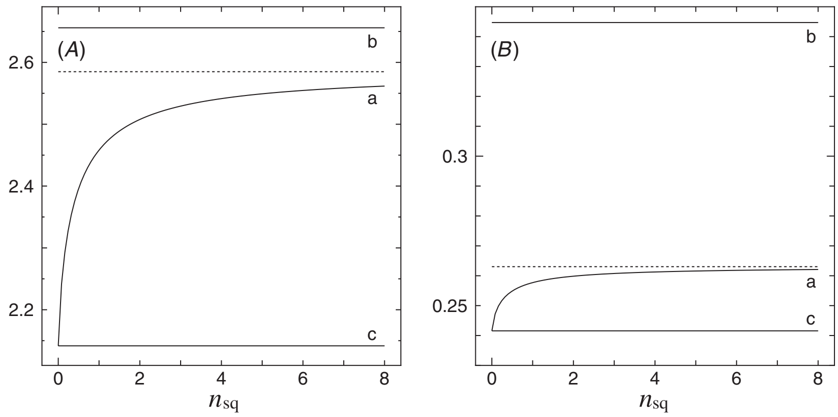

Figure 4. The dependence of the quantum dense coding capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$ on the squeezed photon number $\bar{n}_{sq}$, where $C_D(\hat{\mathcal{L}}_{\bar{n}})$ is given by a, $C_E(\hat{\mathcal{L}}_{\bar{n}})$ by b, $C_H(\hat{\mathcal{L}}_{\bar{n}})$ by c, and the dotted line represents equation (60). We set $\bar{m} = 5$ and $\bar{n} = 1$ in (*A*) and $\bar{m} = 1$ and $\bar{n} = 5$ in (*B*). In the figures, the information is measured in bits.

becomes identical with the Holevo capacity $C_H(\hat{\mathcal{L}}_{\bar{n}})$. On the other hand, if the condition $\bar{n}_{\mathrm{sq}} \gg \bar{n}\bar{n}_{\mathrm{sq}} \gg 1$ is fulfilled, the quantum dense coding capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$ becomes

$$C_D(\hat{\mathcal{L}}_{\bar{n}}) = \ln(\bar{m} + \bar{n}) - \ln \bar{n} = \ln\left(1 + \frac{\bar{m}}{\bar{n}}\right) \tag{60}$$

which is equal to the classical capacity derived from the Shannon information theory. The capacity $C_D(\hat{\mathcal{L}}_{\bar{n}})$ of the quantum dense coding as the function of the squeezed photon number $\bar{n}_{\mathrm{sq}}$ is plotted in figure 4. It is found from the figure that the entanglement-assisted capacity is more effective when the input signal power is weak and the channel noisy is large.

In the quantum dense coding of continuous variables, we have assumed that the input power $\bar{m}$ is the average value of $|\alpha|^2$, where α is the amplitude of the displacement operator $\hat{D}(\alpha)$ which the sender applies to the mode in the two-mode squeezed-vacuum state with the finite squeezing energy. This means that the total power of the system at the sender side is given by $\bar{m} + \bar{n}_{\mathrm{sq}}$. Thus there is the case that the input power is $\bar{M} = \bar{m} + \bar{n}_{\mathrm{sq}}$ and the optimum sharing of $\bar{M}$ should be considered. Such a problem will be discussed elsewhere.

4. Summary

In this paper, we have obtained the entanglement-assisted capacity, the Holevo capacity and the quantum dense coding capacity of the thermalizing quantum channel. Since the thermalizing quantum channel is Gaussian, it is enough to consider the Gaussian prior probability distribution and the Gaussian input state in the optimization procedure. Hence the analytic expressions of these capacities can be obtained. It has been found from the result that the entanglement-assisted capacity becomes much greater than any other capacity when the channel noise is large and the input signal power is small. This is reasonable because the sender and the receiver can use the quantum entanglement unlimitedly in the case of the entanglement-assisted capacity. The thermalizing quantum channel not only appears in the communication systems with random noise of a signal amplitude and the quantum teleportation channel, but also has a simple structure so that the several quantities related to the channel can be obtained analytically. Therefore, the thermalizing quantum channel is suitable for examining the general theory of quantum communication.

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